

ON TIGHTNESS OF THE SKEW RANDOM WALKS

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ABSTRACT. The primary purpose of this article is to prove a tightness of α -skew random walks. The tightness result implies, in particular, that the α -skew Brownian motion can be constructed as the scaling limit of such random walks. Our proof of tightness is based on a forth-order moment method.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Skew Brownian motion was introduced by Itô and McKean [5] to furnish a construction of certain stochastic processes related to Feller's classification of second order differential operators associated with diffusion processes (see also Section 4.2 in [6]). For $\alpha \in (0, 1)$, the α -skew Brownian motion is defined as a one-dimensional Markov process with the same transition mechanism as of the usual Brownian motion, with the only exception that the excursions away from zero are assigned a positive sign with probability α and a negative sign with probability $1 - \alpha$. The signs form an i.i.d. sequence and are chosen independently of the past history of the process. If $\alpha = 1/2$, the process is the usual Brownian motion.

Formally, the α -skew random walk on \mathbb{Z} starting at 0 is defined as the birth-death Markov chain $S^{(\alpha)} = \{S_k^{(\alpha)}; k \geq 0\}$ with $S_0^\alpha = 0$ and one-step transition probabilities given by

$$\begin{aligned} P(S_{k+1}^{(\alpha)} = m + 1 | S_k^{(\alpha)} = m) &= \begin{cases} \alpha & \text{if } m = 0 \\ \frac{1}{2} & \text{otherwise.} \end{cases} \\ P(S_{k+1}^{(\alpha)} = m - 1 | S_k^{(\alpha)} = m) &= \begin{cases} 1 - \alpha & \text{if } m = 0 \\ \frac{1}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

In the special case $\alpha = \frac{1}{2}$, $S^{(\frac{1}{2})}$ is a simple symmetric random walk on \mathbb{Z} . Notice that when $\alpha \neq 1/2$, the jumps (in general, increments) of the random walk are not independent.

Harrison and Shepp [3] asserted (without proof) that the functional central limit theorem for reflecting Brownian motion can be used to construct skew Brownian motion as the limiting process of a suitably modified symmetric random walk on the integer lattice. This result has served as a foundation for numerical algorithms tracking moving particle in a highly heterogeneous porous media; see, for instance, [4, 7, 8, 9]. In [7] it was suggested that tightness could be obtained based on second moments, however this is not possible even in the case of simple symmetric random walk. The lack of statistical independence of the increments make a fourth moment proof all the more challenging. Although proofs of FCLTs in more general frameworks have subsequently been obtained by other methods, e.g., by Skorokhod

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embedding in [10], a self-contained simple proof of tightness for simple skew random walk has not been available in the literature.

The main goal of this paper is to prove the following result. Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions from $\mathbb{R}_+ = [0, \infty)$ into \mathbb{R} , equipped with the topology of uniform convergence on compact sets. For $n \in \mathbb{N}$, let $X_n^{(\alpha)} \in C(\mathbb{R}_+, \mathbb{R})$ denote the following linear interpolation of $S_{[nt]}^{(\alpha)}$:

$$X_n^{(\alpha)}(t) = \frac{1}{\sqrt{n}} \left(S_{[nt]}^{(\alpha)} + (nt - [nt]) \cdot S_{[nt]+1}^{(\alpha)} \right).$$

Here and henceforth $[x]$ denotes the integer part of a real number x .

Theorem 1.1. *For any $\alpha \in (0, 1)$, there exists a constant $C_\alpha > 0$, such that the inequality*

$$E|X_n^{(\alpha)}(t) - X_n^{(\alpha)}(s)|^4 \leq C_\alpha |s - t|^2,$$

holds uniformly for all $s, t > 0$, and $n \in \mathbb{N}$.

The results stated above implies (see, for instance, [2, p. 98]):

Corollary 1.2. *The family of processes $X_n^{(\alpha)}$, $n \in \mathbb{N}$, is tight in $C(\mathbb{R}_+, \mathbb{R})$.*

The rest of the paper is organized as follows. In Section 2 we define α -skew random walks and review some of their basic properties. Theorem 1.1 and the tightness property for the skew random walks are proved in Section 2.

2. PROOF OF THEOREM 1.1

In this section we complete the proof of our main result, Theorem 1.1. In what follows we will use S to denote the simple symmetric random walk $S^{(\frac{1}{2})}$. The following observations can be found in [3].

Proposition 2.1.

- (a) $|S^{(\alpha)}|$ has the same distribution as $|S|$ on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. That is, $|S^{(\alpha)}|$ is a simple symmetric random walk on \mathbb{Z}_+ , reflected at 0.
- (b) The processes $-S^{(\alpha)}$ and $S^{(1-\alpha)}$ have the same distribution.

The next statement describes n -step transition probabilities of the skew random walks by relating them to those of S (see, for instance, [7, p. 436]).

Proposition 2.2. *For $m \in \mathbb{Z}$, $k > 0$*

$$P(S_k^{(\alpha)} = m) = \begin{cases} \alpha \cdot P(|S_k| = m) & \text{if } m > 0 \\ (1 - \alpha) \cdot P(|S_k| = -m) & \text{if } m < 0 \\ P(|S_k^{(\alpha)}| = 0) = P(|S_k| = 0) & \text{if } m = 0 \end{cases}$$

The following observation is evident from the explicit form of the distribution function of $S_k^{(\alpha)}$, given in Proposition 2.2.

Proposition 2.3. *With probability one,*

$$E(S_{j+1}^{(\alpha)} - S_j^{(\alpha)} | S_j^{(\alpha)}) = (2\alpha - 1) \mathbf{1}_{\{S_j^{(\alpha)} = 0\}}$$

and

$$E[(S_{i+1}^{(\alpha)} - S_i^{(\alpha)})^2 | S_i^{(\alpha)}] = 1,$$

To show the result of Theorem 1.1, we will need a corollary to Karamata's Tauberian theorem, which we are going now to state. For a measure μ on $[0, \infty)$, denote by $\hat{\mu}(\lambda) := \int_0^\infty e^{-\lambda x} \mu(dx)$ the Laplace transform of μ . The transform is well-defined for $\lambda \in (c, \infty)$, where $c > 0$ is a non-negative constant, possibly $+\infty$. If μ and ν are measures on $[0, \infty)$ such that $\hat{\mu}(\lambda)$ and $\hat{\nu}(\lambda)$ both exist for all $\lambda > 0$, then the convolution $\gamma = \mu * \nu$ has the Laplace transform $\hat{\gamma}(\lambda) = \hat{\mu}(\lambda)\hat{\nu}(\lambda)$ for $\lambda > 0$. If μ is a discrete measure concentrated on \mathbb{Z}_+ , one can identify μ with a sequence μ_n of its values on $n \in \mathbb{Z}_+$. For such discrete measures, we have (see, e.g., Corollary 8.10 in [1, p. 118]).

Proposition 2.4. *Let $\tilde{\mu}(t) = \sum_{n=0}^\infty \mu_n t^n$, $0 \leq t < 1$, where $\{\mu_n\}_{n=0}^\infty$ is a sequence of non-negative numbers. For L slowly varying at infinity and $0 \leq \theta < \infty$ one has*

$$\tilde{\mu}(t) \sim (1-t)^{-\theta} L\left(\frac{1}{1-t}\right) \text{ as } t \uparrow 1$$

if and only if

$$\sum_{j=0}^n \mu_j \sim \frac{1}{\Gamma(\theta)} n^\theta L(n) \text{ as } n \rightarrow \infty.$$

Here and henceforth, $a_n \sim b_n$ for two sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ means $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

We are now in a position to prove the following key proposition. Define a sequence $\{g(k)\}_{k \in \mathbb{Z}_+}$ as follows

$$g(k) = \begin{cases} 0 & \text{if } k \in \mathbb{N} \text{ is odd} \\ \binom{2i}{i} 2^{-2i} & \text{if } k = 2i \in \mathbb{N} \text{ is even.} \end{cases}$$

Note that in view of Proposition 2.2,

$$g(k) = P(S_k = 0) = P(|S_k| = 0) = P(|S_k^{(\alpha)}| = 0) = P(S_k^{(\alpha)} = 0).$$

Proposition 2.5.

- (a) *If $\mu(j) = g * g(j)$ then $\sum_{j=0}^m \mu(j) \sim m$*
- (b) *If $\nu(j) = g * g * g * g(j)$ then $\sum_{j=0}^m \nu(j) \sim m^2$.*

Proof. For $t \in (0, 1)$, let $\tilde{g}(t) = \sum_{k=0}^\infty g(k)t^k$. Notice that $\tilde{g}(t)$ is well-defined since $g(k) = P(S_k = 0) < 1$ for $k \geq 0$. Since $g(2j) = \binom{2j}{j} 2^{-2j} = (-1)^j \binom{-\frac{1}{2}}{j}$, we have

$$\begin{aligned} \tilde{g}(t) &= \sum_{k=0}^\infty g(k)t^k = \sum_{j=0}^\infty \binom{2j}{j} 2^{-2j} t^{2j} = \sum_{j=0}^\infty (-1)^j \binom{-\frac{1}{2}}{j} t^{2j} \\ &= \sum_{j=0}^\infty \binom{-\frac{1}{2}}{j} (-t^2)^j = (1-t^2)^{-\frac{1}{2}}. \end{aligned}$$

Notice that, using the notation of Proposition 2.4, $\tilde{g}(t) = \hat{g}(\lambda)$ if $t = e^{-\lambda}$. Therefore, $\tilde{\mu}(t) = \tilde{g}^2(t) = (1-t^2)^{-1}$ while $\tilde{\nu}(t) = \tilde{g}^4(t) = (1-t^2)^{-2}$. Thus claims (a) and (b) of the proposition follow from Proposition 2.4 applied, respectively, with $\theta = 1$, $L = 1$ for μ and with $\theta = 2$, $L = 1$ for ν . \square

The last technical lemma we need is the following claim.

Lemma 2.6. *For integers $0 < i_1 < i_2 < i_3 < i_4$ define*

$$A(i_1, i_2, i_3) := E(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}) (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}),$$

and

$$B(i_1, i_2, i_3, i_4) := E(S_{i_4+1}^{(\alpha)} - S_{i_4}^{(\alpha)}) (S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)}) (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}) (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}).$$

Then there is a constant $C > 0$ such that

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq k-j} A(i_1, i_2, i_3) \leq C|k-j|^2,$$

and

$$\sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq k-j} B(i_1, i_2, i_3, i_4) \leq C|k-j|^2.$$

Proof. Using Proposition 2.3, the Markov property, and the fact the excursions of $S^{(\alpha)}$ away from zero are the same as excursions of the simple symmetric random walk S , we obtain

$$\begin{aligned} A(i_1, i_2, i_3) &= E(S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}) (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)} \mathbf{1}_{\{S_{i_1}^{(\alpha)}=0\}} \mathbf{1}_{\{S_{i_2}^{(\alpha)}=0\}}) \\ &= P(S_{i_1} = 0) \cdot (2\alpha - 1) \cdot P(S_{i_2} = 0 | S_{i_1} = 0) \cdot (2\alpha - 1) \\ &= (2\alpha - 1)^2 g(i_1) g(i_2 - i_1). \end{aligned}$$

Therefore,

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq k-j} A(i_1, i_2, i_3) \leq \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2-1} g(i_2 - i_1) g(i_1).$$

Using Proposition 2.5, we obtain

$$\begin{aligned} \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} \sum_{i_1=0}^{i_2-1} g(i_2 - i_1) g(i_1) &= \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{i_3-1} g * g(i_2) \leq \sum_{i_3=0}^{[k-j]} \sum_{i_2=0}^{[k-j]} g * g(i_2) \\ &\leq C_1 |k-j|^2, \end{aligned}$$

for some constant $C_1 > 0$ and any $k, j \in \mathbb{N}$.

Similarly,

$$\begin{aligned} B(i_1, i_2, i_3, i_4) &= (2\alpha - 1)^4 \cdot P(S_{i_1} = 0) \cdot \prod_{a=1}^3 P(S_{i_a+1} = 0 | S_{i_a} = 0) \\ &= (2\alpha - 1)^4 g(i_1) g(i_2 - i_1) g(i_3 - i_2) g(i_4 - i_3). \end{aligned}$$

Hence, using again Proposition 2.5,

$$\sum_{0 \leq i_1 < i_2 < i_3 < i_4} B(i_1, i_2, i_3, i_4) \leq \sum_{i_4=0}^{[k-j]} g * g * g * g(i_4) \leq C_2 |k-j|^2,$$

for some constant $C_2 > 0$ and any $k, j \in \mathbb{N}$. □

We are now in a position to complete the proof of our main result.

Completion of the proof of Theorem 1.1.

First consider the case where $s = \frac{j}{n} < \frac{k}{n} = t$ are grid points. Then

$$\begin{aligned}
E \left| \frac{S_{[nt]}^{(\alpha)}}{\sqrt{n}} - \frac{S_{[ns]}^{(\alpha)}}{\sqrt{n}} \right|^4 &= \frac{1}{n^2} E \left| S_k^{(\alpha)} - S_j^{(\alpha)} \right|^4 = \frac{1}{n^2} E \left| \sum_{i=j}^{k-1} (S_{i+1}^{(\alpha)} - S_i^{(\alpha)}) \right|^4 \\
&= \frac{1}{n^2} \sum_{i=j}^{k-1} E (S_{i+1}^{(\alpha)} - S_i^{(\alpha)})^4 + \frac{1}{n^2} \sum_{i_1 < i_2 \leq k-j} E (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)})^2 \\
&\quad + \frac{1}{n^2} \sum_{i_1 < i_2 < i_3 \leq k-j} E (S_{i_3+1}^{(\alpha)} - S_{i_3}^{(\alpha)})^2 (S_{i_2+1}^{(\alpha)} - S_{i_2}^{(\alpha)}) (S_{i_1+1}^{(\alpha)} - S_{i_1}^{(\alpha)}) \\
&\quad + \frac{1}{n^2} \sum_{i_1 < i_2 < i_3 < i_4 \leq k-j} \prod_{a=1}^4 E (S_{i_a+1}^{(\alpha)} - S_{i_a}^{(\alpha)}) \\
&\leq \frac{1}{n^2} \sum_{i=j}^{k-1} 1 + \frac{1}{n^2} \binom{k-j}{2} \binom{k-j}{2} + \frac{1}{n^2} C_1 |k-j|^2 + \frac{1}{n^2} C_2 |k-j|^2 \\
&\leq C_3 |t-s|^2,
\end{aligned}$$

for a large enough constant $C_3 > 0$.

To conclude the proof of Theorem 1.1, it remains to observe that for non-grid points s and t one can use an approximation by neighbor grid points. In fact, the approximation argument given in [2, pp. 100-101] for regular random walks goes through verbatim. \square

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